Periodic solutions and stability of a fifth order difference equation

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Abstract

In this paper, we study the periodic solutions of difference equation $x_{n+1} = x_{n-2}x_{n-4} - 1$, $n = 0, 1, ...$ where the initial conditions are real numbers. Moreover, we handle eventually periodic solutions with period two. We also investigate the stability of this difference equation.

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1 Introduction

Difference equations or their systems have a huge interest among the researchers. This interest related to applications of these equations or systems. There are many applications of difference equations in many fields of science. There exist many articles related to our paper as follows:

In [8], Kent et al. studied the periodicity and boundedness of solutions of difference equation

$$x_{n+1} = x_n x_{n-1} - 1.$$

They also investigated long term behaviors of solutions of related difference equation. Furthermore, in [23], Wang et al. handled convergence of solutions of related difference equation. Moreover, in [11], Liu et al. studied some properties of solutions of related difference equation.

In [9], Kent et al. examined the boundedness and periodicity of solutions of difference equation

$$x_{n+1} = x_{n-1}x_{n-2} - 1.$$

In [10], Kent et al. studied the stability, periodicity and boundedness of solutions of difference equation

$$x_{n+1} = x_n x_{n-2} - 1.$$

In [7], Kent et al. handled the periodicity, asymptotic periodicity, unbounded solutions and local stability of difference equation

$$x_{n+1} = x_n x_{n-3} - 1.$$

In [18], Taşdemir et al. studied the stability, periodicity, bounded and unbounded solutions of the difference equation

$$x_{n+1} = x_{n-1}x_{n-3} - 1.$$

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Moreover, in [19], the authors investigated convergence of negative equilibrium of related difference equation.

In [20], Taşdemir et al. considered the periodicity, asymptotic periodicity, stability of solutions of difference equation

$$x_{n+1} = x_{n-2}x_{n-3} - 1.$$

In [21], Taşdemir et al. handled the stability, periodicity and asymptotic periodic solutions of difference equation

$$x_{n+1} = x_{n-1}x_{n-4} - 1.$$

In [22], Taşdemir et al. investigated the periodicity, eventually periodicity and stability analysis of difference equation

$$x_{n+1} = x_{n-3}x_{n-4} - 1.$$

In this paper, we study the dynamics of solutions of the following difference equation

$$x_{n+1} = x_{n-2}x_{n-4} - 1, n = 0, 1, ... \tag{1.1}$$

where the initial conditions are real numbers. We especially investigate the stability and periodicity of solutions of difference equation (1.1). We also overcome the eventually periodic solutions with period two.

Note that, Eq.(1.1) be a member of the class of equations of the form

$$x_{n+1} = x_{n-l}x_{n-k} - 1, n = 0, 1, ... \tag{1.2}$$

with special choices of l and k, where $k, l \in \mathbb{N}_0$ and $l < k$. In literature, there are many papers and books related to difference equations (see [1]–[23]).

Now, we present some important definitions and theorems.

Definition 1.1. Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $(k + 1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \quad n = 0, 1, \cdots. \tag{1.3}$$

A solution of Eq.(1.3) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(1.3) for all $n \geq -k$. As a special case of Eq.(1.3), for every set of initial conditions $x_0, x_{-1}, x_{-2}, x_{-3}, x_{-4} \in I$, the fifth order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}), \quad n = 0, 1, \cdots, \tag{1.4}$$

has a unique solution $\{x_n\}_{n=-4}^{\infty}$.

Definition 1.2. A solution of Eq.(1.3) that is constant for all $n \geq -k$ is called an equilibrium solution of Eq.(1.3). If

$$x_n = \overline{x}, \text{ for all } n \geq -k$$

is an equilibrium solution of Eq.(1.3), then $\overline{x}$ is called an equilibrium point, or simply an equilibrium of Eq.(1.3). So a point $\overline{x} \in I$ is called an equilibrium point of Eq.(1.3) if

$$\overline{x} = f(\overline{x}, \overline{x}, \cdots, \overline{x}),$$

that is,

$$x_n = \overline{x} \quad \text{for} \quad n \geq -k$$

is a solution of Eq.(1.3).

Definition 1.3. Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}\left(\bar{x}, \bar{x}, \cdots, \bar{x}\right), \text{ for } i = 0, 1, 2, \cdots, k$$

denote the partial deriative of $f\left(u_0, u_1, \cdots, u_k\right)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(1.3).

The equation

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \cdots + q_k z_{n-k}, \ k = 0, 1, \cdots, \tag{1.5}$$

is called the linearized equation of Eq.(1.3) about the equilibrium point $\bar{x}$.

Definition 1.4. The equation

$$\lambda^{k+1} - q_0 \lambda^k - q_1 \lambda^{k-1} - \cdots - q_{k-1}\lambda - q_k = 0 \tag{1.6}$$

is called the characteristic equation of Eq.(1.5) about $\bar{x}$.

Definition 1.5. Let $\overline{x}$ an equilibrium point of Eq.(1.3).

(a) An equilibrium point $\overline{x}$ of Eq.(1.3) is called locally stable if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.3) with

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

then

$$|x_n - \overline{x}| < \varepsilon, \quad \text{for all } n \geq -k.$$

(b) An equilibrium point $\overline{x}$ of Eq.(1.3) is called locally asymptotically stable if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1.3) with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

then we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(c) An equilibrium point $\overline{x}$ of Eq.(1.3) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.3), we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(d) An equilibrium point $\overline{x}$ of Eq.(1.3) is called globally asymptotically stable if it is locally stable, and a global attractor.

(e) An equilibrium point $\overline{x}$ of Eq.(1.3) is called unstable if it is not locally stable.

Theorem 1.6. Assume that the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\overline{x}$. Then the following statements are true:

(a) When all the roots of Eq.(1.6) have absolute value less than one, then the equilibrium point $\overline{x}$ of Eq.(1.3) is locally asymptotically stable. Moreover, in this here the equilibrium point $\overline{x}$ of Eq.(1.3) is called sink.

(b) If at least one root of Eq.(2.3) has absolute value greater than one, then the equilibrium point $\overline{x}$ of Eq.(1.3) is unstable.

Definition 1.7. A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.3) is called periodic with period p if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n, \text{ for all } n \geq -k.$$

2 Stability analysis of Eq.(1.1)

In this section, we find the equilibrium points of Eq.(1.1). Then, we examine the characteristic equation of Eq.(1.1). Lastly, we investigate the stability of Eq.(1.1).

Lemma 2.1. There are two equilibrium points of Eq.(1.1), namely,

$$\bar{x}_1 = \frac{1+\sqrt{5}}{2}, \bar{x}_2 = \frac{1-\sqrt{5}}{2}. \tag{2.1}$$

Proof. Let $x_n = \bar{x}$ for all $n \geq -4$. Therefore, we get from Eq.(1.1)

$$\bar{x} = \bar{x} \cdot \bar{x} - 1.$$

Q.E.D.

Lemma 2.2. Assume that $\bar{x}$ is an equilibrium point of Eq.(1.1). Hence the linearized equation of Eq.(1.1) is

$$z_{n+1} - \overline{x} \cdot z_{n-2} - \overline{x} \cdot z_{n-4} = 0. \tag{2.2}$$

Proof. Let I be some interval of real numbers and let $f : I^5 \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}) = x_{n-2}x_{n-4} - 1.$$

Then, we obtain the followings:

$$q_0 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = [0](\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = 0,$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = [x_{n-4}](\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = 0,$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = [0](\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = \overline{x},$$

$$q_3 = \frac{\partial f}{\partial x_{n-3}}(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = [0](\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = 0,$$

$$q_4 = \frac{\partial f}{\partial x_{n-4}}(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = [x_{n-1}](\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{x}) = \bar{x}.$$

Therefore, the linearized equation associated with Eq.(1.1) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 \cdot z_n + q_1 \cdot z_{n-1} + q_2 \cdot z_{n-2} + q_3 \cdot z_{n-3} + q_4 \cdot z_{n-4}$$

and so $z_{n+1} - \overline{x} \cdot z_{n-2} - \overline{x} \cdot z_{n-4} = 0$. Q.E.D.

Lemma 2.3. The characteristic equation of Eq.(1.1) about its equilibrium point $\overline{x}$ is

$$\lambda^5 - \overline{x} \cdot \lambda^2 - \overline{x} = 0. \tag{2.3}$$

Now, we examine the stability of equilibrium points of Eq.(1.1).

Theorem 2.4. The positive equilibrium $\bar{x}_1$ of Eq.(1.1) is unstable.

Proof. We take (2.3) with $\bar{x}_1 = \frac{1+\sqrt{5}}{2}$. Then, we get the following characteristic equation

$$\lambda^5 - \left(\frac{1+\sqrt{5}}{2}\right)\lambda^2 - \left(\frac{1+\sqrt{5}}{2}\right) = 0. \tag{2.4}$$

Thus, we obtain five roots of (2.4) as follows:

$$\begin{aligned} \lambda_1 &\approx 1.35669, \\ \lambda_{2,3} &\approx -0.817695 \pm 0.913305i, \\ \lambda_{4,5} &\approx 0.139352 \pm 0.879896i. \end{aligned}$$

Hence, we have,

$$|\lambda_1| > |\lambda_{2,3}| > 1 > |\lambda_{4,5}|.$$

Consequently, the equilibrium $\bar{x}_1$ of Eq.(1.1) is unstable. Q.E.D.

Theorem 2.5. The negative equilibrium $\bar{x}_2$ of Eq.(1.1) is unstable.

Proof. We consider (2.3) with $\bar{x}_2 = \frac{1-\sqrt{5}}{2}$. Then, we obtain the following characteristic equation

$$\lambda^5 - \left(\frac{1-\sqrt{5}}{2}\right)\lambda^2 - \left(\frac{1-\sqrt{5}}{2}\right) = 0. \tag{2.5}$$

Hence, we have five roots of (2.5) as follows:

$$\begin{aligned} \lambda_1 &\approx -1.054797, \\ \lambda_{2,3} &\approx -0.163078 \pm 0.772937i, \\ \lambda_{4,5} &\approx 0.690472 \pm 0.679856i. \end{aligned}$$

Therefore, we get

$$|\lambda_1| > 1 > |\lambda_{2,3}| > |\lambda_{4,5}|.$$

So, the equilibrium $\bar{x}_2$ of Eq.(1.1) is unstable. Q.E.D.

3 Periodic solutions of Eq.(1.1)

Now, we study the periodic solutions of Eq.(1.1) with period two and three.

Theorem 3.1. There are two periodic solutions of Eq.(1.1).

Proof. Let a, b be real numbers with $a \neq b$. Suppose that $\{x_{2n}\}_{n=-2}^{\infty} = a$ and $\{x_{2n-1}\}_{n=-1}^{\infty} = b$. Then, from Eq.(1.1), we obtain

$$b = a^2 - 1, \quad (3.1)$$
$$a = b^2 - 1. \quad (3.2)$$

From (3.1) and (3.2), we have the following four cases:

$$\begin{aligned} \textit{Case i. } a &= b = \bar{x}_1, \\ \textit{Case ii. } a &= b = \bar{x}_2, \\ \textit{Case iii. } a &= 0, b = -1, \\ \textit{Case iv. } a &= -1, b = 0. \end{aligned}$$

Since the cases *i.* and *ii.* are trivial solutions, they are not periodic solutions. Other two cases are periodic solutions with period two. Q.E.D.

Theorem 3.2. Eq.(1.1) has eventually two periodic solutions as following forms:

Case i. $\{x_n\}_{n=-4}^{\infty} = \{\cdots, x_N, x_{N+1}, x_{N+2}, x_{N+3}, x_{N+4}, 0, -1, 0, -1, \cdots\}$, where $x_{N+2}x_N = 1$, $x_{N+3}x_{N+1} = 0$ and $x_{N+4}x_{N+2} = 1$.

Case ii. $\{x_n\}_{n=-4}^{\infty} = \{\cdots, x_N, x_{N+1}, x_{N+2}, x_{N+3}, x_{N+4}, -1, 0, -1, 0, \cdots\}$, where $x_{N+2}x_N = 0$, $x_{N+4}x_{N+2} = 0$ and $x_{N+3} = x_{N+1} = -1$.

Proof. Firstly, we consider proof of Case i. Let $\{x_n\}_{n=-4}^{\infty}$ be a eventually two periodic solution of Eq.(1.1). Hence, we have $x_{N+5} = 0, x_{N+6} = -1, x_{N+7} = 0$ and $x_{N+8} = 0$. Thus, we obtain the followings:

$$\begin{aligned} x_{N+5} &= x_{N+2}x_N - 1 = 0 \Rightarrow x_{N+2}x_N = 1, \\ x_{N+6} &= x_{N+3}x_{N+1} - 1 = -1 \Rightarrow x_{N+3}x_{N+1} = 0, \\ x_{N+7} &= x_{N+4}x_{N+2} - 1 = 0 \Rightarrow x_{N+4}x_{N+2} = 1, \\ x_{N+8} &= x_{N+5}x_{N+3} - 1 = -1 \Rightarrow x_{N+5}x_{N+3} = 0. \end{aligned}$$

This completes the proof of Case i. The proof of Case ii is similar to proof of Case i. Q.E.D.

Theorem 3.3. There are no three periodic solutions of Eq.(1.1).

Proof. Proof. Let a, b, c be real numbers such that at least two of them are different. Assume that $\{x_{3n}\}_{n=-1}^{\infty} = a$, $\{x_{3n-1}\}_{n=-1}^{\infty} = b$ and $\{x_{3n-2}\}_{n=0}^{\infty} = c$. Then, from Eq.(1.1), we get

$$a = a \cdot c - 1, \quad (3.3)$$
$$b = b \cdot a - 1, \quad (3.4)$$
$$c = c \cdot b - 1. \quad (3.5)$$

From (3.3) - (3.5), we have the following two cases:

$$Case\ i.\ a = b = c = \bar{x}_1, \tag{3.6}$$
$$Case\ ii.\ a = b = c = \bar{x}_2. \tag{3.7}$$

Then, (3.6) and (3.7) are not three periodic solutions, because these are trivial solutions of Eq.(1.1). This completes the proof. Q.E.D.

4 Numerical simulations

In this section, we present two numerical examples which verifies our theoretical results.

Example 4.1. Consider Eq.(1.1) with the initial conditions $x_{-4} = -1$, $x_{-3} = 0$, $x_{-2} = -1$, $x_{-1} = 0$ and $x_0 = -1$. Then Eq.(1.1) has two periodic solution. The Figure 1 shows the first 80 terms of Eq.(1.1).

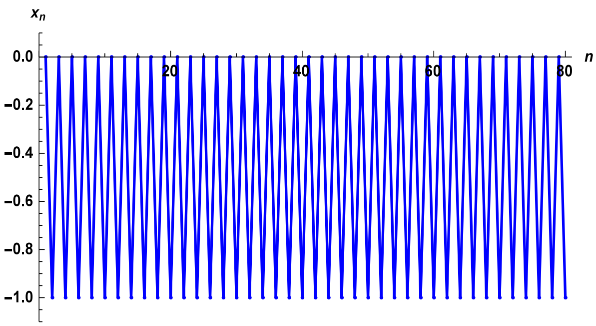

FIGURE 1. Plot of Eq.(1.1).

Example 4.2. Consider Eq.(1.1) with the initial conditions $x_{-4} = -\frac{15}{112}$, $x_{-3} = \frac{1}{4}$, $x_{-2} = \frac{28}{3}$, $x_{-1} = 12$ and $x_0 = \frac{3}{4}$. Then Eq.(1.1) has the eventually periodic solution with period two. The Figure 2 shows the first 80 terms of Eq.(1.1).

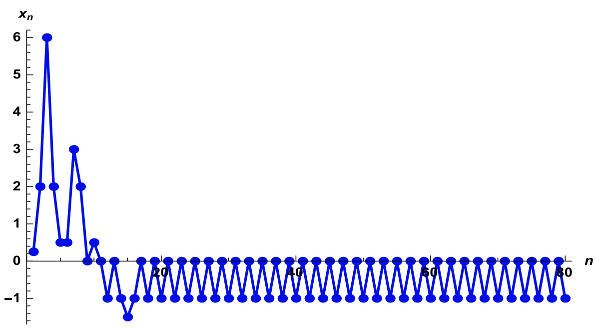

FIGURE 2. Plot of Eq.(1.1).

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